The fundamental functional equation

Let π be an irreducible admissible rep of $G = GL_2(F)$, F a p-adic field.

Recall that we have Kirillov and Whittaker models

$$\mathcal{K}(\pi) = \left\{ \begin{array}{l} \xi : F^\times \to \mathbb{C} \text{ locally constant,} \\ \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi(x) = \tau(bx)\xi(ax) \end{array} \right\}$$

$$\mathcal{W}(\pi) \subset \left\{ \begin{array}{l} w : G \to \mathbb{C} \text{ locally constant,} \\ w\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \tau(x) w(g) \end{array} \right\}$$

$$w(hg) = \pi(g) w(h)$$

$$\mathcal{K}(\pi) \xrightarrow{\cong} \mathcal{W}(\pi)$$

$$\xi \longmapsto w_\xi : G \to \mathbb{C}$$

$$w_\xi(g) = \pi(g)\xi(1)$$

$$\xi_w : F^\times \to \mathbb{C} \longleftarrow w$$

$$\xi_w(x) = w\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Given $W \in \mathcal{W}(\pi)$ and a character χ of $F^\times$, set

$$L_W(g;\chi,s) = \int_{F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(x)^{-1} |x|^{2s-1} d^\times x$$

Theorem 8:

(a) $L_W(g;\chi,s)$ converges for $\mathrm{Re}(s) >> 0$

(b) $L_W(g;\chi,s)$ can be analytically continued to a meromorphic function with ≤ 2 poles.

(c) $\exists$ meromorphic $\gamma_\pi(\chi,s)$ such that

FE $\quad L_W(\varpi g; \omega_\pi - \chi, 1-s) = \gamma_\pi(\chi,s) L_W(g;\chi,s)$

1 $\quad \gamma_\pi(\omega_\pi - \chi, 1-s)\gamma_\pi(\chi,s) = \omega_\pi(-1)$

(Here $\gamma_\pi(\chi,s)$ is independent of W and g, ω_π is the central character of π, and for some reason we're using additive notation $\omega_\pi - \chi$ for the character $\omega_\pi \chi^{-1}$ of $F^\times$.

Also $\quad \varpi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.)

Note that for any $g \in G$ we have

$$L_{\pi(g)W}(e;\chi,s) = L_W(g;\chi,s)$$

Letting $\xi = \xi_W \in \mathcal{K}(\pi)$ and

$$M_\xi(\chi,s) = \int_{F^\times} \xi(x)\chi(x)^{-1}|x|^{2s-1}d^\times x,$$

Theorem 8 is equivalent to

(a) $M_\xi(\chi,s)$ converges for $\mathrm{Re}(s) \gg 0$

(b) $M_\xi(\chi,s)$ can be analytically continued to a meromorphic function with ≤ 2 poles.

(c) $\exists$ meromorphic $\gamma_\pi(\chi,s)$ satisfying

FE' $\quad M_{\pi(w)\xi}(\omega_\pi\chi^{-1},1-s) = \gamma_\pi(\chi,s)M_\xi(\chi,s)$

1 $\quad \gamma_\pi(\omega_\pi\chi^{-1},1-s)\,\gamma_\pi(\chi,s) = \omega_\pi(-1)$

For (a) and (b), recall the result from last time:

$\boxed{f_1, f_2 \in \mathcal{S}(F)}$

π	$\mathcal{K}(\pi)$
supercuspidal	$\mathcal{S}(F^\times)$
principal π_{μ_1,μ_2} $(\mu_1 \neq \mu_2)$	$\lvert x\rvert^{1/2}\left[\mu_1(x) f_1(x) + \mu_2(x) f_2(x)\right]$
principal π_{μ_1,μ_2} $(\mu_1 = \mu_2)$	$\lvert x\rvert^{1/2}\left[\mu_1(x) f_1(x) + \mu_2(x) v(x) f_2(x)\right]$
special π_{μ_1,μ_2} $(\mu_1\mu_2^{-1} = \lvert\,\rvert)$	$\lvert x\rvert^{1/2}\mu_1(x) f_1(x)$
special π_{μ_1,μ_2} $(\mu_1\mu_2^{-1} = \lvert\,\rvert^{-1})$	$\lvert x\rvert^{1/2}\mu_2(x) f_2(x)$

So if π is supercuspidal, then $\xi \in \mathcal{S}(F^\times)$ and

$$M_\xi(\chi, s) = \int_{F^\times} \xi(x)\,\chi(x)^{-1}\,\lvert x\rvert^{2s-1}\,d^\times x$$

converges for all $s \in \mathbb{C}$ to an analytic function.

Otherwise, $M_\xi(\chi, s)$ can be written as the sum of at most two integrals, each of which is of one of the two types

$$\int_{F^\times} f(x)\lambda(x)\,|x|^{2s}\,d^\times x$$

or

$$\int_{F^\times} f(x)\lambda(x)\sigma(x)|x|^{2s}\,d^\times x$$

with $f \in \mathcal{S}(F)$ and λ a character of $F^\times$.

<u>Claim:</u> Each of these two integrals converges for $\mathrm{Re}(s) \gg 0$ and can be analytically continued to a meromorphic function with at most 1 pole? (See Section 14.)

We now focus on

(c) $\exists$ meromorphic $\gamma_\pi(\chi, s)$ satisfying

(FE') $M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s) = \gamma_\pi(\chi,s)\, M_\xi(\chi,s)$

(1) $\gamma_\pi(\omega_\pi - \chi, 1-s)\,\gamma_\pi(\chi,s) = \omega_\pi(-1)$

Step 1: FE′ holds for $\xi \in \mathcal{S}(F^\times) \subset \mathcal{K}(\pi)$.

Back in Section 3, we saw that $\mathcal{S}(F^\times)$ is spanned by multiplicative translates of functions of the form

$$\xi(x) = \begin{cases} \lambda(x) & \text{if } x \in \mathcal{O}_F^\times \\ 0 & \text{otherwise} \end{cases}$$

with λ a char of $\mathcal{O}_F^\times$

For such ξ, the two zeta functions in FE′ become

$$M_\xi(\chi, s) = \int_{\mathcal{O}_F^\times} \lambda(x)\, \chi^{-1}(x)\, d^\times x$$

$$M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s) = \int_{F^\times} J_\pi(x,\lambda)\, \chi(x)\, |x|^{1-2s}\, d^\times x$$

where $\pi(w)\xi(x) = \omega_\pi(x) J_\pi(x,\lambda)$

By orthogonality of characters, it is clear that the first integral is zero if $\lambda \neq \chi|_{\mathcal{O}_F^\times}$. So is the second integral (use

$$J_\pi(xu,\lambda) = J_\pi(x,\lambda)\, \lambda^{-1}(u) \text{ for } u \in \mathcal{O}_F^\times.)$$

So we are left with the case $\lambda = \chi\big|_{\mathcal{O}_F^\times}$.

The first integral is then

$$M_\xi(\chi, s) = \int_{\mathcal{O}_F^\times} d^\times x \;=1$$ if Haar measure $d^\times x$ is normalised appropriately.

Set then, for any $\xi \in \mathcal{S}(F^\times)$,

$$\gamma_\pi(\chi, s) = \int_{F^\times} J_\pi(x, \chi)\, \chi(x)\, |x|^{1-2s} d^\times x$$

<u>Step 2:</u> The identity (1) holds for $\gamma_\pi(\chi, s)$

In the proof that $\dim\left(\mathcal{K}(\pi)/\mathcal{S}(F^\times)\right) < \infty$, Godement showed the existence of $0 \neq \xi \in \mathcal{S}(F^\times) \cap \pi(w)\mathcal{S}(F^\times)$ such that

$$\xi(xu) = \xi(x)\chi(u) \qquad \forall u \in \mathcal{O}_F^\times$$

Take such ξ, then

$$M_\xi(\chi, s) = \sum_{n\in\mathbb{Z}} \xi(\varpi^n)\,\chi(\varpi^n)^{-1}\,|\varpi^n|^{2s-1}$$

where the sum is finite and (since $\xi \neq 0$) nonempty.
We apply (FE') to ξ twice:

$$\omega_\pi(-1) M_\xi(\chi, s) = M_{\pi(-1)\xi}(\chi, s)$$

$$= M_{\pi(w)\pi(w)\xi}(\chi, s)$$

$$= \gamma_\pi(\omega_\pi - \chi, 1-s)\, M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s)$$

$$= \gamma_\pi(\omega_\pi - \chi, 1-s)\, \gamma_\pi(\chi, s)\, M_\xi(\chi, s)$$

Since $M_\xi(\chi, s)$ is not identically zero, we conclude that (1) holds.

Step 3: (FE') is true for all $\xi \in \mathcal{K}(\pi)$.

Write $\xi = \xi_1 + \pi(w)\xi_2$, $\xi_1, \xi_2 \in \mathcal{S}(F^\times)$.

Then

$$M_{\pi(\omega)\xi}(\omega_\pi - \chi, 1-s)$$

$$= M_{\pi(\omega)\xi_1}(\omega_\pi - \chi, 1-s) + \omega_\pi(-1) M_{\xi_2}(\omega_\pi - \chi, 1-s)$$

$$= \gamma_\pi(\chi, s) M_{\xi_1}(\chi, s) + \gamma_\pi(\omega_\pi - \chi, 1-s)\gamma_\pi(\chi, s) M_{\xi_2}\begin{pmatrix}\omega_\pi - \chi, \\ 1-s\end{pmatrix}$$

$$= \gamma_\pi(\chi, s)\left(M_{\xi_1}(\chi, s) + M_{\pi(\omega)\xi_2}(\chi, s)\right)$$

$$= \gamma_\pi(\chi, s)\, M_{\xi}(\chi, s)$$